LIBERTY PAPER SET

STD. 12 : Mathematics

Full Solution

Time : 3 Hours **ASSIGNMENT PAPER 3**

PART A

1. (C) **2.** (B) **3.** (C) **4.** (D) **5.** (A) **6.** (B) **7.** (C) **8.** (B) **9.** (D) **10.** (D) **11.** (A) **12.** (A) **13.** (A) **14.** (A) **15.** (D) **16.** (C) **17.** (B) **18.** (B) **19.** (D) **20.** (A) **21.** (C) **22.** (C) **23.** (C) **24.** (B) **25.** (C) **26.** (B) **27.** (B) **28.** (C) **29.** (D) **30.** (D) **31.** (B) **32.** (B) **33.** (C) **34.** (A) **35.** (B) **36.** (D) **37.** (B) **38.** (D) **39.** (A) **40.** (C) **41.** (A) **42.** (B) **43.** (D) **44.** (A) **45.** (B) **46.** (A) **47.** (C) **48.** (C) **49.** (D) **50.** (A)

PART B

SECTION A

1.

Method 1 :

Here, $tan^{-1}\left(\frac{cosx}{1-sinx}\right)$

$$= tan^{-1}\left[\frac{cos^2\frac{x}{2}-sin^2\frac{x}{2}}{cos^2\frac{x}{2}+sin^2\frac{x}{2}-2sin\frac{x}{2}\cdot cos\frac{x}{2}}\right]$$

$$= tan^{-1}\left[\frac{\left(cos\frac{x}{2}+sin\frac{x}{2}\right)\left(cos\frac{x}{2}-sin\frac{x}{2}\right)}{\left(cos\frac{x}{2}-sin\frac{x}{2}\right)^2}\right]$$

$$= tan^{-1}\left[\frac{cos\frac{x}{2}+sin\frac{x}{2}}{cos\frac{x}{2}-sin\frac{x}{2}}\right]$$

($\because$ Dividing numerator & denominator by $cos\frac{\pi}{2}$)

$$= tan^{-1}\left[\frac{1+tan\frac{x}{2}}{1-tan\frac{x}{2}}\right]$$

$$= tan^{-1}\left[tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$$

$$\left(\begin{aligned}&\because -\frac{3\pi}{2}<x<\frac{\pi}{2}\\&\Rightarrow -\frac{3\pi}{4}<\frac{x}{2}<\frac{\pi}{4}\\&\Rightarrow -\frac{\pi}{2}<\frac{\pi}{4}+\frac{x}{2}<\frac{\pi}{2}\\&\Rightarrow \left(\frac{\pi}{4}+\frac{x}{2}\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \quad ...(1)\end{aligned}\right)$$

$$= \frac{\pi}{4}+\frac{x}{2}$$ ($\because$ From equation (1))

Method 2 :

$$tan^{-1}\left(\frac{cosx}{1-sinx}\right) = tan^{-1}\left[\frac{sin\left(\frac{\pi}{2}-x\right)}{1-cos\left(\frac{\pi}{2}-x\right)}\right]$$

$$= tan^{-1}\left[\frac{sin\left(\frac{\pi-2x}{2}\right)}{1-cos\left(\frac{\pi-2x}{2}\right)}\right]$$

$$= tan^{-1}\left[\frac{2sin\left(\frac{\pi-2x}{4}\right)cos\left(\frac{\pi-2x}{4}\right)}{2sin^2\left(\frac{\pi-2x}{4}\right)}\right]$$

$$= tan^{-1}\left[cot\left(\frac{\pi-2x}{4}\right)\right]$$

$$= tan^{-1}\left[tan\left(\frac{\pi}{2}-\frac{\pi-2x}{4}\right)\right]$$

$$= tan^{-1}\left[tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$$

$$\left(\begin{aligned}&\because -\frac{3\pi}{2}<x<\frac{\pi}{2}\\&\Rightarrow -\frac{3\pi}{4}<\frac{x}{2}<\frac{\pi}{4}\\&\Rightarrow -\frac{\pi}{2}<\frac{\pi}{4}+\frac{x}{2}<\frac{\pi}{2}\\&\Rightarrow \left(\frac{\pi}{4}+\frac{x}{2}\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \quad ...(1)\end{aligned}\right)$$

$$= \frac{\pi}{4}+\frac{x}{2}$$ ($\because$ From equation (1))

2.

L.H.S. $= sin^{-1}(2x\sqrt{1-x^2})$

Suppose, $x = cos\,\theta$

$\therefore\ \theta = cos^{-1}x,\ \theta \in [0, \pi]$

$= sin^{-1}(2\,cos\,\theta\sqrt{1-cos^2\theta})$

$= sin^{-1}(2\,cos\,\theta\,sin\,\theta)$

$= sin^{-1}(sin\,2\theta)$

Here, $\frac{1}{\sqrt{2}} \le x \le 1$

$\therefore \cos\frac{\pi}{4} \le \cos\theta \le \cos 0$

$\therefore 0 \le \theta \le \frac{\pi}{4}$ ($\because \cos\theta$ is decreasing function in first quadrant)

$\therefore 0 \le 2\theta \le \frac{\pi}{2}$

$\therefore 2\theta \in \left[0, \frac{\pi}{2}\right] \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ .... (1)

$\therefore \sin^{-1}(\sin 2\theta) = 2\theta$ ($\because$ From equation (1))

$= 2\cos^{-1}x$

= R.H.S.

3.

$y = 3\cos(\log x) + 4\sin(\log x)$

Differentiate w.r.t. x,

$y_1 = -3\sin[\log x]\cdot\frac{1}{x} + 4\cos[\log x]\cdot\frac{1}{x}$

$\therefore xy_1 = -3\sin(\log x) + 4\cos(\log x)$

Now, differentiate again w.r.t. x,

$x\cdot y_2 + y_1 = -3\cos(\log x)\cdot\frac{1}{x} + 4(-\sin(\log x))\cdot\frac{1}{x}$

$\therefore x^2y_2 + xy_1 = -(3\cos(\log x) + 4\sin(\log x))$

$\therefore x^2y_2 + xy_1 = -y$

$\therefore x^2y_2 + xy_1 + y = 0$

4.

Method 1 :

Derivative of $\sqrt{x}$ is $\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

Thus, we use the substitution $\sqrt{x} = t$.

So, that, $\frac{1}{2\sqrt{x}}dx = dt$.

$\therefore dx = 2t\,dt$

Thus,

$$\int \frac{\tan^4\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}dx = \int \frac{2t\tan^4 t\sec^2 t\,dt}{t}$$

$$= 2\int \tan^4 t\sec^2 t\,dt$$

Again, we make another substitution, $\tan t = u$

So, that $\sec^2 t\,dt = du$

$\therefore 2\int \tan^4 t\sec^2 t\,dt = 2\int u^4 du$

$= 2\cdot\frac{u^5}{5} + c$

$= \frac{2}{5}\tan^5 t + c$ $(u = \tan t)$

$= \frac{2}{5}\tan^5\sqrt{x} + c$ $(t = \sqrt{x})$

Hence, $\int \frac{\tan^4\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}dx = \frac{2}{5}\tan^5\sqrt{x} + c$

Method 2 :

$$\int \frac{\tan^4\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}dx$$

Take, $\tan\sqrt{x} = u$,

$\therefore \sec^2\sqrt{x}\,\frac{1}{2\sqrt{x}}dx = du$

$\therefore \frac{\sec^2\sqrt{x}}{\sqrt{x}}dx = 2\,du$

$= 2\int u^4 du$

$= \frac{2u^5}{5} + c$

$= \frac{2\tan^5\sqrt{x}}{5} + c$

5.

Required Area,

A = 4|I|

$\therefore I = \int_0^{\frac{\pi}{2}} y\,dx$

$\therefore I = \int_0^{\frac{\pi}{2}} \sin x\,dx$

$\therefore I = [-\cos x]_0^{\frac{\pi}{2}}$

$\therefore I = -\cos\frac{\pi}{2} + \cos 0$

$\therefore I = 1$

Now, A = 4|I|

= 4|1|

$\therefore$ A = 4 sq. units.

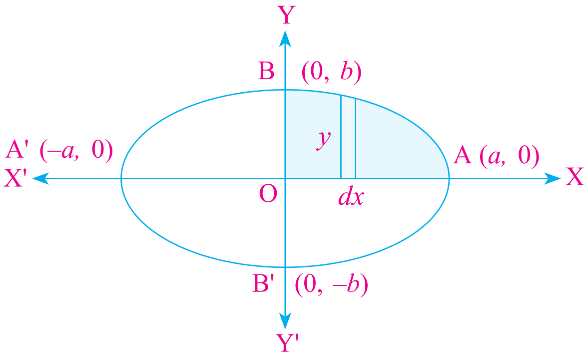
6.

From Fig, the area of the region ABA'B'A bounded by the ellipse = 4 × area of the region AOBA the first quadrant bounded by the curve x-axis and the ordinates $x = 0, x = a$) (as the ellipse is symmetrical about both x-axis and y-axis)

$= 4\int_0^a y$ (taking verticalstrips)

Now, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$

but as the region AOBA lies in the first quadrant, y is taken as positive. So, the required area is,



$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}sin^{-1}\frac{x}{a}\right]_0^a$

$= \frac{4b}{a}\left[\left(\frac{a}{2}\times 0+\frac{a^2}{2}sin^{-1}1\right)-(0)\right]$

$= \frac{4b}{a}\,\frac{a^2}{2}\,\frac{\pi}{2}$

$= \pi ab$ sq. unit

7.

$sin\ x\ cos\ y\ dx + cos\ x\ sin\ y\ dy = 0$

$\therefore\ cos\ x\ sin\ y\ dy = -sin\ x\ cos\ y\ dx$

$\therefore\ \frac{sin\ y}{cos\ y}\,dy = -\frac{sin\ x}{cos\ x}\,dx$

$\therefore\ tan\ y\ dy = -\ tan\ x\ dx$

→ Integrate both the sides,

$\therefore\ \int tan\ y\ dy = -\int tan\ x\ dx$

$\therefore\ log\ |sec\ y| = -log\ |sec\ x| + log\ |c|$

$\therefore\ log\ |sec\ y| + log\ |sec\ x| = log\ |c|$

$\therefore\ log\ |sec\ x \cdot sec\ y| = log\ |c|$

$\therefore\ sec\ x \cdot sec\ y = c$ ... (1)

→ Curve passes through the point $\left(0,\ \frac{\pi}{4}\right)$

$\therefore\ sec\ 0 \cdot sec\ \frac{\pi}{4} = c$

$\therefore\ c = \sqrt{2}$

→ Put the value of c in equation (1),

$sec\ x\ sec\ y = \sqrt{2}$

$\therefore\ \frac{sec\ x}{\sqrt{2}} = \frac{1}{sec\ y}$

$\therefore\ \frac{sec\ x}{\sqrt{2}} = cos\ y;$

Which is required equation of curve.

8.

We have,

$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$= \vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$

$= |\vec{a}|^2-2(\vec{a}\cdot\vec{b})+|\vec{b}|^2$

$= (2)^2-2(4)+(3)^2$

$= 5$

$\therefore\ |\vec{a}-\vec{b}| = \sqrt{5}$

9.

$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$

Now, $\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$

Direction of line $\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

Vector from, $\vec{r} = \vec{a} + \lambda\vec{b}\ \ \lambda \in R$

$\therefore\ \vec{r} = (5\hat{i}-4\hat{j}+6\hat{k})+\lambda(3\hat{i}+7\hat{j}+2\hat{k}), \lambda \in R$

Which is required vector equation of line.

10.

Suppose, A(1, –1, 2), B(3, 4, –2), C(0, 3, 2), D(3, 5, 6) are given points.

$\vec{b_1} = \overrightarrow{AB}$

= Position vector of B – Position vector of A

$= 2\hat{i} + 5\hat{j} - 4\hat{k}$

$\vec{b_2} = \overrightarrow{CD}$

= Position vector of D – Position vector of C

$= 3\hat{i} + 2\hat{j} + 4\hat{k}$

$\vec{b_1}\cdot\vec{b_2} = (2\hat{i} + 5\hat{j} - 4\hat{k})\cdot(3\hat{i} + 2\hat{j} + 4\hat{k})$

$= 6 + 10 - 16$

$= 0$

$\therefore\ \vec{b_1}$ and $\vec{b_2}$ are perpendicular to each other.

Therefore, given lines are perpendicular to each other.

11.

$P(A) = \frac{1}{2}$, $P(B) = P$, $P(A \cup B) = \frac{3}{5}$

(i) A and B are mutually exclusive events.

$\therefore\ P(A \cap B) = 0$

$\therefore\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore\ \frac{3}{5} = \frac{1}{2} + P - 0$

$\therefore\ P = \frac{3}{5} - \frac{1}{2}$

$\therefore\ P = \frac{1}{10}$

(ii) A and B are independent events.

$\therefore$ P(A $\cap$ B) = P(A) · P(B)

$\therefore$ P(A $\cup$ B) = P(A) + P(B) – P(A $\cap$ B)

$\therefore \frac{3}{5} = \frac{1}{2} + P - \frac{1}{2}P$

$\therefore \frac{3}{5} - \frac{1}{2} = \frac{1}{2}P$

$\therefore \frac{1}{2}P = \frac{1}{10}$

$\therefore P = \frac{1}{5}$

12.

$P(A) = \frac{1}{2}$, $P(A) \cdot P(B) = \frac{7}{24}$, $P(B) = \frac{7}{12}$

P(not A or not B) = P(A' $\cup$ B')

$\therefore \frac{1}{4} = P(A \cap B)'$

$= 1 - P(A \cap B)$

$\therefore P(A \cap B) = 1 - \frac{1}{4}$

$\therefore P(A \cap B) = \frac{3}{4}$

$\neq \frac{7}{24}$

$\neq P(A) \cdot P(B)$

$\therefore$ A and B are not independent events.

SECTION B

13.

Option 1 : $x \geq 0$, $f(x) = \frac{x}{1+x} \geqslant 0 \quad |x| = x$

$\forall x_1, x_2 > 0, f(x_1) = f(x_2)$

$\therefore \frac{x_1}{1+x_1} = \frac{x_2}{1+x_2}$

$\therefore x_1 + x_1x_2 = x_2 + x_1x_2$

$\therefore x_1 = x_2$

Option 2 : $x < 0$, $f(x) = \frac{x}{1-x}$ $\quad (\because |x| = -x)$

$\therefore x_1, x_2 < 0, f(x_1) = f(x_2)$

$\therefore \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}$

$\therefore x_1 - x_1x_2 = x_2 - x_1x_2$

$\therefore x_1 = x_2$

Option 3 : $\forall x_1, x_2 \in R$, $x_1 \geq 0$ and

Take, $x_2 < 0$

$f(x_1) = f(x_2)$

$\frac{x_1}{1+x_1} = \frac{x_2}{1-x_2}$

$\therefore x_1 - x_1x_2 = x_2 + x_1x_2$

$\therefore x_1 - x_2 = 2x_1x_2$ which is not possible.

$(\because x_1 \geq 0, x_2 < 0 \; 2x_1x_2 \leq 0)$

when $x_1 - x_2 > 0$

This option is not possible.

$\therefore$ For other two options,

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

$\forall y \in (-1, 1)$

(i) $-1 < y < 0$ then

Suppose, $f(x) = y \; \therefore f(x) < 0$

$\therefore \frac{x}{1-x} = y$

$\therefore x = y - yx$

$\therefore x + yx = y$

$\therefore x(1 + y) = y$

$\therefore x = \frac{y}{1+y} \in R$

Now, $f(x) = f\left(\frac{y}{1+y}\right)$

$= \frac{\frac{y}{1+y}}{1 - \frac{y}{1+y}}$

$= \frac{y}{1+y-y} = y$

(ii) $0 \leq y < 1$ then

Suppose, $f(x) = y \; \therefore f(x) \geq 0$

$\therefore \frac{x}{1+x} = y$

$\therefore x = y + yx$

$\therefore x - yx = y$

$\therefore x(1 - y) = y$

$\therefore x = \frac{y}{1-y} \in R$

Now $f(x) = f\left(\frac{y}{1-y}\right)$

$= \frac{\frac{y}{1-y}}{1 + \frac{y}{1-y}}$

$= \frac{y}{1-y+y}$

$= y$

$\therefore$ For two option, f is onto function.

14.

Suppose A is symmetric matrix.

$\therefore$ A' = A ... (1)

Take, X = B'AB

X' = (B'AB)'

= (B'(AB))' ($\because$ Associative law)

= (AB)' (B')'

= (B'A') B

= B'(A'B) ($\because$ Associative law)

= B' (AB) ($\because$ From equation (1))

= B'AB

= X

$\therefore$ X is symmetric matrix.

$\therefore$ B'AB is symmetric matrix.

Suppose, A is skew symmetric matrix.

$\therefore$ A' = – A ... (2)

Take, X = B'AB

X' = (B'AB)'

= (B'(AB))'

= (AB)' (B')'

= (B'A') B

= (–B'A)B ($\because$ From equation (2))

= – (B'A)B

= – B'AB

X' = – X

$\therefore$ X is skew symmetric matrix.

$\therefore$ B'AB is skew symmetric matrix.

15.

For finding B^{-1},

$$|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$= 1(3-0) - 2(-1-0) - 2(2-0)$

$= 3 + 2 - 4$

$= 1 \neq 0$

$\therefore B^{-1}$ exists.

For finding *adj* B,

Co-factor of element 1 $A_{11} = (-1)^2 \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix}$

$= 1(3-0)$

$= 3$

Co-factor of element 2 $A_{12} = (-1)^3 \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$

$= (-1)(-1-0)$

$= 1$

Co-factor of element –2 $A_{13} = (-1)^4 \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix}$

$= 1(2-0)$

$= 2$

Co-factor of element –1 $A_{21} = (-1)^3 \begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix}$

$= (-1)(2-4)$

$= 2$

Co-factor of element 3 $A_{22} = (-1)^4 \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}$

$= 1(1-0)$

$= 1$

Co-factor of element 0 $A_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix}$

$= (-1)(-2-0)$

$= 2$

Co-factor of element 0 $A_{31} = (-1)^4 \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix}$

$= 1(0+6)$

$= 6$

Co-factor of element –2 $A_{32} = (-1)^5 \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix}$

$= (-1)(0-2)$

$= 2$

Co-factor of element 1 $A_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$

$= 1(3+2)$

$= 5$

$$adj\ B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|}\ adj\ B = \frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\therefore B^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$(AB)^{-1} = B^{-1} A^{-1}$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

16.

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$\therefore\ x\sqrt{1+y} = -y\sqrt{1+x}$

Now, squaring both sides,

$x^2(1+y) = y^2(1+x)$

$\therefore\quad x^2 + x^2y = y^2 + y^2x$

$\therefore\quad x^2 - y^2 + x^2y - y^2x = 0$

$\therefore\quad (x-y)(x+y) + (x-y)xy = 0$

$\therefore\quad (x-y)(x+y+xy) = 0$

$x - y = 0$ fu $x + y + xy = 0$

When, $x = y$

$x\sqrt{1+x} + x\sqrt{1+x} = 0$

$\therefore\quad 2x\sqrt{1+x} = 0$

$\therefore\quad \sqrt{1+x} = 0$

Thus, $\dfrac{dy}{dx}$ does not exist.

$\therefore\quad x + y + xy = 0$

$\therefore\quad x + y(1+x) = 0$

$\therefore\quad y(1+x) = -x$

$\therefore\quad y = \dfrac{-x}{1+x}$

Now, differentiate w.r.t. x,

$$\frac{dy}{dx} = -\left[\frac{(1+x)(1)-(x)}{(1+x)^2}\right]$$

$$\frac{dy}{dx} = \frac{-1}{(1+x)^2}$$

17.

Here, $f(x) = (x+1)^3(x-3)^3$

$f'(x) = 3(x+1)^2(x-3)^3 + 3(x-3)^2(x+1)^3$

$= 3(x+1)^2(x-3)^2(x-3+x+1)$

$= 3(x+1)^2(x-3)^2(2x-2)$

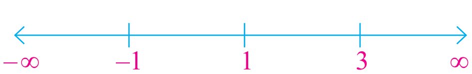
$= 6(x+1)^2(x-3)^2(x-1)$

$\rightarrow$ For finding intervals,

$f'(x) = 0$

$\therefore\ 6(x+1)^2(x-3)^2(x-1) = 0$

$\therefore\ (x+1)^2 = 0$	$x - 3 = 0$	$x - 1 = 0$
$\therefore\ x + 1 = 0$	$x = 3$	$x = 1$
$x = -1$		



$\rightarrow\ \forall x \in (-\infty, -1)$ $\Rightarrow (x+1)^2 > 0, (x-3)^2 > 0, x-1 < 0$

$\Rightarrow (x+1)^2(x-3)^2(x-1) < 0$

$\Rightarrow 6(x+1)^2(x-3)^2(x-1) < 0$

$\Rightarrow f'(x) < 0$

$\therefore f$ is strictly decreasing function in the interval of $(-\infty, -1)$

$\rightarrow\ \forall x \in (-1, 1)$ $\Rightarrow (x+1)^2 > 0, (x-3)^2 > 0, x-1 < 0$

$\Rightarrow (x+1)^2(x-3)^2(x-1) < 0$

$\Rightarrow 6(x+1)^2(x-3)^2(x-1) < 0$

$\Rightarrow f'(x) < 0$

$\therefore f$ is strictly decreasing function in the interval of $(-1, 1)$

$\rightarrow\ \forall x \in (1, 3)$ $\Rightarrow (x+1)^2 > 0, (x-3)^2 > 0, x-1 > 0$

$\Rightarrow (x+1)^2(x-3)^2(x-1) > 0$

$\Rightarrow 6(x+1)^2(x-3)^2(x-1) > 0$

$\Rightarrow f'(x) > 0$

$\therefore f$ is strictly decreasing function in the interval of $(1, 3)$

$\rightarrow\ \forall x \in (3, \infty)$ $\Rightarrow (x+1)^2 > 0, (x-3)^2 > 0, x-1 > 0$

$\Rightarrow (x+1)^2(x-3)^2(x-1) > 0$

$\Rightarrow 6(x+1)^2(x-3)^2(x-1) > 0$

$\Rightarrow f'(x) > 0$

$\therefore f$ is strictly decreasing function in the interval of $(3, \infty)$

18.

Two adjacent sides of Parellelogram ABCD

$\overrightarrow{AB} = \vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$

and $\overrightarrow{BC} = \vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$

Diagonal vector $\overrightarrow{AC} = \vec{c}$

$= \vec{a} + \vec{b}$

$= 3\hat{i} - 6\hat{j} + 2\hat{k}$

$|\vec{c}| = \sqrt{9+36+4}$

$= \sqrt{49}$

$|\vec{c}| = 7$

Unit vector parellel to diagonal vector $\vec{c}$

$$= \frac{\vec{c}}{|\vec{c}|} = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$$

In this way, find parellel vector to the diagonal $\overrightarrow{BD}$

Area of Parellelogram,

$\Delta = |\vec{a} \times \vec{b}|$ ... (1)

Now, $\vec{a} \times \vec{b} = (2\hat{i} - 4\hat{j} + 5\hat{k}) \times (\hat{i} - 2\hat{j} - 3\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$

$\therefore\ \vec{a} \times \vec{b} = 22\hat{i} + 11\hat{j} + 0\hat{k}$

Area of parallelogram $= \sqrt{(22)^2 + (11)^2}$

$\therefore\ |\vec{a} \times \vec{b}| = \sqrt{484 + 121}$

$= \sqrt{605}$

$= 11\sqrt{5}$

From equation (1),

$\Delta = 11\sqrt{5}$ sq. units.

19.

L : $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$;

M : $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

$\therefore \ \vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}$

$\therefore \ \vec{b_1} = \hat{i} - \hat{j} + \hat{k}$, and

$\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}$;

$\vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$

Now, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$

$= -3\hat{i} + 0\hat{j} + 3\hat{k}$

$\neq \vec{0}$

$\therefore$ Lines are intersecting lines or skew lines.

$\vec{a_2} - \vec{a_1} = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k})$

$= \hat{i} - 3\hat{j} - 2\hat{k}$

Now, $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})$

$= (\hat{i} - 3\hat{j} - 2\hat{k}) \cdot (-3\hat{i} + 0\hat{j} + 3\hat{k})$

$= -3 + 0 - 6$

$= -9$

$\neq 0$

$\therefore$ Lines are skew line.

Shortest distance between two lines,

$$= \frac{\left|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})\right|}{\vec{b_1} \times \vec{b_2}}$$

$$= \frac{|-9|}{\sqrt{9+0+9}}$$

$$= \frac{9}{\sqrt{18}}$$

$$= \frac{9}{3\sqrt{2}}$$

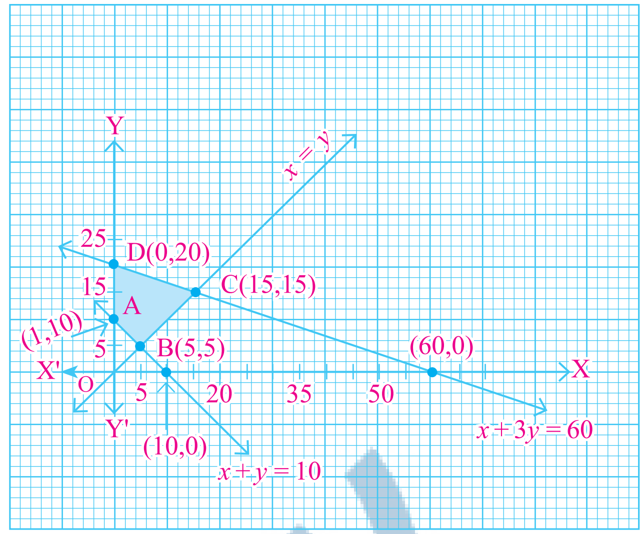
$$= \frac{3}{\sqrt{2}}$$ yuf{

20.

First of all, let us graph the feasible region of the system of linear inequalities (2) to (5).

The feasible region ABCD is shown in the Fig 12.4. Note that the region is bounded.

The coordinates of the corner points A, B, C and D are (0, 10), (5, 5), (15,15) and (0, 20) respectively.



Corner Point	Corresponding value of $Z = 3x + 9y$
A(0, 10)	90
B(5, 5)	60 $\rightarrow$ Minimum
C(15, 15)	180 } $\rightarrow$ Maximum
D(0, 20)	180 } (Multiple optimal solutions)

We now find the minimum and maximum value of Z. From the table, we find that

the minimum value of Z is 60 at the point B (5, 5) of the feasible region.

The maximum value of Z on the feasible region occurs at the two corner points

C (15, 15) and D (0, 20) and it is 180 in each case.

Note : Observe that in the above example, the problem has multiple optimal solutions at the corner points C and D, i.e. the both points produce same maximum value 180. In such cases, you can see that every point on the line segment CD joining the two corner points C and D also give the same maximum value. Same is also true in the case if the two points produce same minimum value.

21.

Let A be the event that the machine produces 2 acceptable items.

Also let B_1 represent the event of correct set up and B represent the event of incorrect setup.

Now, $P(B_1) = 0.8$,

$P(B_2) = 0.2$

$P(A \mid B_1) = 0.9 \times 0.9$ and

$P(A \mid B_2) = 0.4 \times 0.4$

Therefore,

$$P(B_1 \mid A) = \frac{P(B_1)P(A \mid B_1)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2)}$$

$$= \frac{0.8 \times 0.9 \times 0.9}{0.8 \times 0.9 \times 0.9 + 0.2 \times 0.4 \times 0.4}$$

$$= \frac{648}{680}$$

$= 0.95$

SECTION C

22.

Here, $B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$

Suppose, $P = \frac{1}{2}(B + B')$

$$= \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix}$$

Now, $P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$

Thus, $P = \frac{1}{2}(B + B')$ is symmetric matrix.

Also, Let,

$$Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Then, $Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{2} \\ \frac{-1}{2} & 0 & -3 \\ \frac{-5}{2} & 3 & 0 \end{bmatrix} = -Q.$

Thus, $Q = \frac{1}{2}(B - B')$ is skew symmetric matrix.

Now, $P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$$

23.

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$$

$= 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2)$

$= 0 + 3(-2) + 5(1)$

$= -6 + 5$

$= -1 \neq 0$

$\therefore$ A^{-1} exists.

For finding *adj* A,

Co-factor of element 2 $A_{11} = (-1)^2 \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix}$

$= 1(-4 + 4)$

$= 0$

Co-factor of element –3 $A_{12} = (-1)^3 \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix}$

$= (-1)(-6 + 4)$

$= 2$

Co-factor of element 5 $A_{13} = (-1)^4 \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$

$= 1(3 - 2)$

$= 1$

Co-factor of element 3 $A_{21} = (-1)^3 \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix}$

$= (-1)(6 - 5)$

$= -1$

Co-factor of element 2 $A_{22} = (-1)^4 \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix}$

$= 1(-4 - 5)$

$= -9$

Co-factor of element –4 $A_{23} = (-1)^5 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix}$

$= (-1)(2 + 3)$

$= -5$

Co-factor of element 1 $A_{31} = (-1)^4 \begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix}$

$= 1(12 - 10)$

$= 2$

Co-factor of element 1 $A_{32} = (-1)^5 \begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix}$

$= (-1)(-8 - 15)$

$= 23$

Co-factor of element –2 $A_{33} = (-1)^6 \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix}$

$= 1(4 + 9)$

$= 13$

$$\therefore\ adj\ A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

$$= \frac{1}{(-1)}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore\ A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Now, $2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

The equation can be represented as matrix form,

$$\therefore \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$\therefore$ AX = B

Where, $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

$\therefore\ X = A^{-1}B$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution : $x = 1, y = 2, z = 3$

24.

Method 1 :

$cosy = xcos(a + y)$

$$\therefore\ x = \frac{cos\,y}{cos(a+y)}$$

Now, Differentiate w.r.t. y,

$$\frac{dx}{dy} = \frac{cos(a+y)\cdot(-siny) - cosy(-sin(a+y))}{[cos(a+y)]^2}$$

$$\therefore\ \frac{dx}{dy} = \frac{-cos(a+y)\cdot siny + cosy \cdot sin(a+y)}{cos^2(a+y)}$$

$$= \frac{sin(a+y-y)}{cos^2(a+y)}$$

$$\frac{dx}{dy} = \frac{sina}{cos^2(a+y)}$$

$$\therefore\ \frac{dy}{dx} = \frac{cos^2(a+y)}{sin\ a}$$

$\left(\begin{array}{l} \because\ cos\ a \neq \pm 1 \\ \therefore\ cos^2\ a \neq 1 \\ \therefore\ 1 - sin^2 a \neq 1 \\ \therefore\ -sin^2 a \neq 0 \\ \therefore\ sin^2 a \neq 0 \\ \therefore\ sin\ a \neq 0 \end{array}\right)$

Method 2 :

$cosy = xcos(a + y)$

Now, Differentiate w.r.t. x,

$$-siny \cdot \frac{dy}{dx} = x \cdot (-sin(a+y))\frac{dy}{dx} + cos(a+y)$$

$$\therefore\ xsin(a+y)\frac{dy}{dx} - siny\frac{dy}{dx} = cos(a+y)$$

$$\therefore\ \frac{dy}{dx}(xsin(a+y) - siny) = cos(a+y)$$

$$\therefore\ \frac{dy}{dx} = \frac{cos(a+y)}{xsin(a+y) - siny} \quad \text{.........(1)}$$

Put, $x = \dfrac{cosy}{cos(a+y)}$ in equation (1),

$$\frac{dy}{dx} = \frac{cos(a+y)}{\frac{cosy}{cos(a+y)}sin(a+y) - siny}$$

$$\frac{dy}{dx} = \frac{cos^2(a+y)}{cosy \cdot sin(a+y) - siny \cdot cos(a+y)}$$

$$= \frac{cos^2(a+y)}{sin(a+y-y)}$$

$$\therefore\ \frac{dy}{dx} = \frac{cos^2(a+y)}{sin\ a}$$

$\left(\begin{array}{l} \because\ cos\ a \neq \pm 1 \\ \therefore\ cos^2\ a \neq 1 \\ \therefore\ 1 - sin^2 a \neq 1 \\ \therefore\ -sin^2 a \neq 0 \\ \therefore\ sin^2 a \neq 0 \\ \therefore\ sin\ a \neq 0 \end{array}\right)$

25.

$$f(x) = \frac{4sinx - 2x - xcosx}{2 + cos\ x}$$

$$= \frac{4sinx - x(2 + cosx)}{2 + cos\ x}$$

$$f(x) = \frac{4\ sin\ x}{2 + cos\ x} - x$$

$$f'(x) = \frac{(2 + cos\ x)(4\ cos\ x) - 4(sinx)(-sinx)}{(2 + cos\ x)^2} - 1$$

$$= \frac{8\ cos\ x + 4\ cos^2 x + 4\ sin^2 x - (2 + cos\ x)^2}{(2 + cos\ x)^2}$$

$$= \frac{8cosx + 4(cos^2 x + sin^2 x) - (4 + 4\ cosx + cos^2 x)}{(2 + cos\ x)^2}$$

$$= \frac{8\ cos\ x + 4 - 4 - 4\ cos\ x - cos^2 x}{(2 + cos\ x)^2}$$

$$f'(x) = \frac{4\cos x - \cos^2 x}{(2+\cos x)^2} \qquad \left(\begin{array}{l}\because\ 4 - \cos x > 0 \\ \Rightarrow (2+\cos x)^2 > 0\end{array}\right)$$

$$f'(x) = \frac{\cos x\,(4-\cos x)}{(2+\cos x)^2}$$

(a) *f* is strictly increasing

$f'(x) > 0$

$\therefore\ \cos x > 0$

$\therefore$ x lies in 1st and 4th quadrant

x lies in 1st quadrant then, $x \in \left(2k\pi, (4k+1)\frac{\pi}{2}\right)$

x lies in 4st quadrant then, $x \in \left((4k+3)\frac{\pi}{2}, (2k+2)\pi\right)$

Note : If $x \in (0, 2\pi)$ then, $x \in \left(0, \frac{\pi}{2}\right)$ OR

$x \in \left(\frac{3\pi}{2}, 2\pi\right)$, f is strictly increasing function.

(b) *f* is strictly decreasing.

$f'(x) < 0$

$\therefore\ \cos x < 0$

$\therefore$ x lies in 2nd and 3rd quadrant,

$x \in \left((4k+1)\frac{\pi}{2},\ (4k+3)\frac{\pi}{2}\right)$

Note : If $x \in (0, 2\pi)$ then, $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

f is decreasing function.

26.

Method-1 :

$$I = \int\left[\log(\log x) + \frac{1}{(\log x)^2}\right]dx$$

$$= \int \log(\log x)\,dx + \int \frac{1}{(\log x)^2}\,dx$$

In the first integral, let us take 1 as the second function. Then integrating it by parts, we get

$$I = x\log(\log x) - \int \frac{1}{x\log x}x\,dx + \int \frac{dx}{(\log x)^2}$$

$$= x\log(\log x) - \int \frac{dx}{\log x} + \int \frac{dx}{(\log x)^2} \quad \text{..... (1)}$$

Again, consider $\int \frac{dx}{\log x}$, take 1 as the second function and integrate it by parts, we have

$$\int \frac{dx}{\log x} = \left[\frac{x}{\log x} - \int x\cdot\left\{\frac{-1}{(\log x)^2}\left(\frac{1}{x}\right)\right\}dx\right]. \quad \text{..... (2)}$$

Putting (2) in (1), we get

$$I = x\log(\log x) - \frac{x}{\log x} - \int \frac{dx}{(\log x)^2} + \int \frac{dx}{(\log x)^2}$$

$$= x\log(\log x) - \frac{x}{\log x} + c$$

Method-2 :

$$I = \int\left(\log(\log x) + \frac{1}{(\log x)^2}\right)dx$$

Take, $\log x = t$

$\therefore\ x = e^t$

$\therefore\ dx = e^t\,dt$

$$= \int\left(\log t + \frac{1}{t^2}\right)e^t\,dt$$

$$= \int e^t\left(\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right)dt$$

$$= \int e^t\left\{\left(\log t - \frac{1}{t}\right) + \left(\frac{1}{t} + \frac{1}{t^2}\right)\right\}dt$$

$$= e^t\left(\log t - \frac{1}{t}\right) + c \qquad \left(\begin{array}{l}\because \int e^x\left(f(x) + f'(x)\right)dx \\ \Rightarrow e^x f(x) + c\end{array}\right)$$

$$= x\left(\log(\log x) - \frac{1}{\log x}\right) + c$$

$$= x\log(\log x) - \frac{x}{\log x} + c$$

27.

$$\frac{dy}{dx} + y\cot x = 4x\,\mathrm{cosec}\,x \quad \text{... (1)}$$

Compare equation (1) with differential equation (1),

$P(x) = \cot x$

$Q(x) = 4x\,\mathrm{cosec}\,x$

$\rightarrow$ Integrating factor I.F. $= e^{\int P(x)\,dx}$

$= e^{\int \cot x\,dx}$

$= e^{\log|\sin x|}$

$= \sin x$

$\rightarrow$ Multiplying equation (1) by $\sin x$,

$$\therefore\ \frac{dy}{dx}\sin x + y\cot x\sin x = 4x\,\mathrm{cosec}\,x\sin x$$

$$\therefore\ \frac{d}{dx}(y\sin x) = 4x$$

$$\therefore\ y\sin x = \int 4x\,dx$$

$$\therefore\ y\sin x = 2x^2 + c \quad \text{... (1)}$$

$\rightarrow$ $y = 0$ when, $x = \frac{\pi}{2}$,

$$\therefore\ 0 = 2\left[\frac{\pi^2}{4}\right] + c$$

$$\therefore\ c = -\frac{\pi^2}{2}$$

$\rightarrow$ Put the value of c in equation (1),

$$\therefore\ y\sin x = 2x^2 - \frac{\pi^2}{2}, \text{ where, } \sin x \neq 0$$

Which is required particular solution of given differential equation.